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LETTER TO THE EDITOR

The adelic components of the index of the Dirac operator

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Abstract. We demonstrate that the index of the Dirac operator can be expressed as an adelic product over the primes.

The arithmetic nature of the expression of the topological indices in the Atiyah–Singer index theorem is striking. They all involve Bernoulli numbers, B_k, which can be related to the Riemann zeta function by the formula

$$-(B_{2k}/2k)=\zeta(1-2k) \tag{1}$$

and the Riemann zeta function can be written as the adelic products of moments of p-adic measures (at least formally, because the factorisations require the real part of $1-2k$ to be greater than 1):

$$\zeta(1-2k)=\prod_p (1-p^{-(1-2k)})^{-1} \tag{2a}$$

$$=\prod_p \int \mathrm{d}\lambda |\lambda|_p^{1-2k} \tag{2b}$$

where the measure $\mathrm{d}\lambda$ is defined by

$$\int_{|\lambda|_p=1} \mathrm{d}\lambda = 1$$

and $|p^L|_p=p^{-L}$ is the p-adic norm‡ [1–4]. One sees [5], for example, that the Todd genus can be written in terms of the polynomial

$$\frac{t}{1-\mathrm{e}^{-t}}=\sum \frac{(-t)^k}{k!} B_k \tag{3a}$$

the $\hat{A}$ genus in terms of

$$\frac{t}{\sinh t}=1-(2^2-2)\frac{B_1}{2!}t^2+(2^4-2)\frac{B_4}{4!}\zeta\ldots \tag{3b}$$

the Hirzebruch signature

$$\frac{\sqrt{t}}{\tanh\sqrt{t}}=1+\tfrac{1}{3}t+\ldots(-1)^{k-1}2^{2k}t^k\frac{B_k}{(2k)!}+\ldots. \tag{3c}$$

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‡ The notion of adeles was introduced in connection with strings in [2–4].

Since each of the expressions can be written in terms of power series with Bernoulli numbers as coefficients, and the Bernoulli number can be expressed as moments of adelic measures, and all of these arithmetic objects conspire to produce an integer (the topological index), it is natural to speculate that there exists a p-adic theory for each prime. In this letter we address the adelic nature of the Dirac index.

Given a Dirac operator $\not{D}$ on a manifold M, the index I of the operator is given by the following formula [5]:

$$I(\not{D}) = \int_{M_n} \prod \frac{\Omega_n/4\pi}{\sinh \Omega_n/4\pi}\bigg|_{\text{top-form}} \tag{4}$$

with Ω_n the eigenvalues of the curvature ∂-form on M. The top-form is the n-form piece with the highest n. Using the Witten index method [6] of calculating $I(\not{D})$ [7-10] we have

$$I(\not{D}) = \mathrm{Tr}(-1)^F \exp(-\beta H) \tag{5a}$$

$$= \int [\mathrm{D}x][\mathrm{D}\psi] \exp\left(-\int_0^\beta L_E\right) \tag{5b}$$

where

$$L_E = \tfrac{1}{2} g_{ij}(\dot{x}^i \dot{x}^j + \psi^i \mathrm{D}_t \psi^j). \tag{6}$$

The free action fixes the functional measure to be

$$[\mathrm{D}x][\mathrm{D}\psi] = \prod_i \mathrm{d}x_0^i \, \mathrm{d}\psi_0^i \prod_{n=1}^{\infty} n \, \mathrm{d}a_n^i \, \mathrm{d}b_n^i \, \mathrm{d}e_n^i \, \mathrm{d}d_n^i \tag{7}$$

when the fields are expressed in Fourier components

$$x^i = x_0^i + \sqrt{\beta} \sum_{n=1}^{\infty} [a_n^i \cos(2\pi n t/\beta) + b_n^i \sin(2\pi n t/\beta)] \tag{8a}$$

$$\psi^i = \left(\frac{\mathrm{i}}{2\pi\beta}\right)^{1/2} \psi_0^i + \sum_{n=0}^{\infty} [c_n^i \cos(2\pi n t/\beta) + d_n^i \sin(2\pi n t/\beta)]. \tag{8b}$$

Choosing a point $x \in M$ as the origin, we may define a system of coordinates in which

$$g_{ij} = \delta_{ij} - \tfrac{1}{3} R_{ikl} x^k x^l. \tag{9}$$

The interaction Lagrangian in normal coordinates is

$$\tfrac{1}{2} g_{ij} \Gamma^j_{kl} \psi^i \dot{x}^k \psi^l = \tfrac{1}{4} R_{ijkl} \psi^i \psi^j x^k \dot{x}^l. \tag{10}$$

The only surviving contribution in the limit $\beta \to 0$ is

$$L_1 = \frac{\mathrm{i}}{8\pi\beta} R_{ijkl} \psi_0^i \psi_0^j x^k \dot{x}^l \tag{11a}$$

$$= \frac{\mathrm{i}}{4\pi\beta} \Omega_{kl} x^k \dot{x}^l \tag{11b}$$

$$\int_0^\beta L_1 = \frac{\mathrm{i}}{4\pi} \Omega_{kl} \sum_{n=1}^{\infty} \pi n (a_n^k b_n^l - a_n^l b_n^k). \tag{12}$$

The most general contribution of these modes to the action is

$$S=\pi n^2(a_n^i b_n^i)\begin{pmatrix}\delta_{ij} & \dfrac{\Omega_i}{4\pi^2 n}\\ \dfrac{-\mathrm{i}\Omega_{ij}}{4\pi^2 n} & \delta_{ij}\end{pmatrix}\begin{pmatrix}a_n^j\\ b_n^j\end{pmatrix}. \tag{13}$$

Integrating over these modes produces a factor of $\det^{-1/2}(1+A)$ where

$$A=\frac{\mathrm{i}}{4\pi^2 n}\begin{pmatrix}0 & \Omega\\ -\Omega & 0\end{pmatrix}. \tag{14}$$

Computing the $\det^{-1/2}(1+A)$ leads to formula (1).

If we consider the sum over the oscillator modes mod p for p prime, we see that $0, p, p^2, \ldots, p^n, \ldots$ are degenerate. Consider then

$$A_p=\frac{|n|_p}{n^{-1}}A \tag{15}$$

where $|\ |_p$ denotes the p-adic norm defined by $|p^L|_p=p^{-L}$. Just like multiplication by n in (11b) on the Fourier components of $x^l(t)$ is implemented by a differential operator, multiplication by $|n|_p$ is implemented by a non-local operator, the p-adic Fourier transform of $|n|_p$. This integral operator can be written as

$$|n|_p-\Gamma(1+\alpha)\int_{Q_p}\mathrm{d}u\,\exp(2\pi\mathrm{i}un)|u|_p^{-2}$$

where we note that it is necessary to extend the domain of integration in the dual variable u to the entire p-adic number field Q_p. This integral is the p-adic version of the Gauss sums that appear in elementary number theory.

We then calculate

$$\det{}^{-1/2}(1+A_p)=1+\tfrac{1}{4}\mathrm{Tr}\,A_p^2+\tfrac{1}{32}(\mathrm{Tr}\,A_p^2)^2+\tfrac{1}{8}\,\mathrm{Tr}\,A_p^4+\ldots \tag{16a}$$

$$=1+\frac{|n|_p^2}{32\pi^4}\mathrm{Tr}\,\Omega^2+\frac{|n|_p^4}{2048\pi^8}[(\mathrm{Tr}\,\Omega^2)^2+2(\mathrm{Tr}\,\Omega^4)]+\ldots. \tag{16b}$$

This formula allows one to localise at a particular prime so that the torsion is associated with that particular prime. As an example we mention the torsion associated with the prime 2 in $\Pi_4\ (\mathrm{SU}(2))=Z_2$. It is localisation at the prime rather than considering the numbers mod p that allows us to see this phenomenon.

We can integrate adelically the product over all primes [7, 8]:

$$\prod_p\int\mathrm{d}\lambda_p\,\det{}^{-1/2}(1+A_p)=1+\frac{\zeta(2)}{32\pi^4}\mathrm{Tr}\,\Omega^2+\frac{\zeta(4)}{2048\pi^2}[(\mathrm{Tr}\,\Omega^2)^2+2(\mathrm{Tr}\,\Omega^4)]+\ldots. \tag{17}$$

This construction can be generalised to the bosonic string by considering the Virasoro subalgebras generated by L_{pn}, L_{-pn} and L_0, for p a prime number, n a positive integer. Each time a new power at p divides n, one obtains a new mode because $|n|_p$ changes. One can also truncate at any power of p to obtain a finite Fourier transform like the one that appears in Gauss sums.

Even more generally than the index of Dirac operator on the space M, one can consider maps of groups into M. The Dirac operator is associated with the circle maps into M with the dual space the integers. The dual space of the group is a lattice. This can be extended to the p-adic integers and then p-adic numbers. On this lattice, the Frobenius map that sends $x \to x^p$ can be represented to pick out the invariant spaces like the odd-dimensional spheres in a Lie group [6]. The simple generalisation, of course, of the loop is just an elliptic curve which has the group structure of $S^1 \times S^1$.

We therefore see that the index of the Dirac operator can be expressed locally at a prime p by considering the oscillator modes mod p.

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References

[1] Gelfand I M, Graev M I and Piatetskii-Shapiro I I 1969 *Representation Theory and Automorphic Forms* (Philadelphia, PA: Saunders)
[2] Grossman B 1987 *Adelic conformal field theory. Preprint* Rockefeller University
[3] Freund P G O and Witten E 1987 *Phys. Lett.* **191B**
[4] Frampton P and Okada Y 1988 *Phys. Rev. Lett.* **60** 484
[5] Atiyah M F and Singer I M 1963 *Bull. Am. Math. Soc.* **69** 42; 1968 *Ann. Math.* **87** 484, 546
[6] Witten E 1982 *Nucl. Phys.* B **202** 661
[7] Alvarez-Gaumé L 1983 *J. Phys. A: Math. Gen.* **16** 4177
[8] Friedan D and Windey P 1984 *Nucl. Phys.* B **235** 395
[9] Goodman M 1986 *Commun. Math. Phys.* **107** 391
[10] Woit P 1988 *On loop space, supersymmetry and the index theorem. Preprint* Harvard University